Calculus 1
Midterm Exam – Solutions
October 3, 2022 (18:30–20:30)

university of groningen

---

1) Prove using the ε-δ definition that $\lim_{x\to 3}(4x-1) = 11$.

Solution. Let $\varepsilon > 0$ be arbitrary and take $\delta = \frac{\varepsilon}{4}$. Then $0 < |x-3| < \delta$ implies that

$$|(4x-1)-11| = |4x-12| = |4(x-3)| = 4|x-3| < 4\delta = \varepsilon.$$

Therefore $\lim_{x\to 3}(4x-1) = 11$.

2) Find the following limits <u>without</u> applying l'Hospital's Rule.

a) $\lim_{x\to -2} \dfrac{x^2-3x-10}{x+2}$

b) $\lim_{x\to 0} \dfrac{\sqrt{\cos(x)}-1}{x^2}$

c) $\lim_{x\to\infty} \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

Solution. a) Factoring the numerator lets us simplify the expression. The limits is then found by the Difference Law and Basic Limits:

$$\lim_{x\to -2} \frac{x^2-3x-10}{x+2} = \lim_{x\to -2} \frac{(x+2)(x-5)}{x+2} = \lim_{x\to -2}(x-5) = -2-5 = -7.$$

b) Using conjugates twice allows for a rewriting of the function as follows

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{\cos(x)}-1}{x^2} &= \lim_{x\to 0} \frac{\sqrt{\cos(x)}-1}{x^2} \cdot \frac{\sqrt{\cos(x)}+1}{\sqrt{\cos(x)}+1} = \lim_{x\to 0} \frac{\cos(x)-1}{x^2(\sqrt{\cos(x)}+1)} \\ &= \lim_{x\to 0} \frac{\cos(x)-1}{x^2(\sqrt{\cos(x)}+1)} \cdot \frac{\cos(x)+1}{\cos(x)+1} = \lim_{x\to 0} \frac{\cos^2(x)-1}{x^2(\sqrt{\cos(x)}+1)(\cos(x)+1)}.\end{aligned}$$

The trigonometric identity

$$\sin^2(x) + \cos^2(x) = 1$$

lets us write the numerator as $(-\sin^2 x)$, hence the limit takes the following form

$$\lim_{x\to 0} \frac{-\sin^2(x)}{x^2(\sqrt{\cos(x)}+1)(\cos(x)+1)}.$$

By using the trigonometric limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(we proved in class) we can evaluate the limit via the Product Law and the continuity of the cosine function at 0. We get

$$\lim_{x\to 0} \frac{-\sin^2(x)}{x^2(\sqrt{\cos(x)}+1)(\cos(x)+1)} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2 \frac{-1}{(\sqrt{\cos(0)}+1)(\cos(0)+1)} = -\frac{1}{4}$$

since $\cos(0) = 1$. Thus we have found that

$$\lim_{x\to 0} \frac{\sqrt{\cos(x)}-1}{x^2} = -\frac{1}{4}.$$

c) Factoring both the numerator and denominator yields

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}} \frac{1 - \dfrac{1}{e^{2x}}}{1 + \dfrac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1,$$

where we used Quotient, Sum, and Difference Laws combined with the fact that $\lim_{x\to\infty} e^{-2x} = 0$.

3) Show using the Squeeze Theorem that $\lim_{x\to 0} \sqrt{x^2} \cos\left(\frac{1}{x^2}\right) = 0$

Solution. For every $x \in \mathbb{R}$ we have $-1 \le \cos(x) \le 1$ and, in particular, this means that

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1, \quad \text{for all } x \in \mathbb{R} \setminus \{0\}.$$

Note that $\sqrt{x^2} \ge 0$ and $\sqrt{x^2} = 0$ if and only if $x = 0$. Hence we have that

$$-\sqrt{x^2} \le \sqrt{x^2} \cos\left(\frac{1}{x^2}\right) \le \sqrt{x^2}, \quad \text{for all } x \in \mathbb{R} \setminus \{0\}.$$

Moreover, we know that $\sqrt{x^2}$, being an elementary function, is continuous on its domain, and in particular at 0, i.e.

$$\lim_{x\to 0} \pm\sqrt{x^2} = 0.$$

Hence by the Squeeze Theorem we can conclude that

$$\lim_{x\to 0} \sqrt{x^2} \cos\left(\frac{1}{x^2}\right) = 0.$$

4) Prove the following statement using mathematical induction on N.

"$(1+x)^N \ge 1 + Nx$ for every integer $N \ge 0$ and real number $x > -1$."

Solution. Step 1 [Base Case]: We prove the base case, i.e., $N = 0$.

$$(1+x)^0 = 1$$

$$1 + 0 \cdot x = 1$$

We have $1 \ge 1$, which is true so the base case holds.

Step 2 [Induction Hypothesis]: We assume that for some integer $k \ge 0$ we have

$$(1+x)^k \ge 1 + kx \tag{1}$$

where $x > -1$.

Step 3 [Inductive Step]: We want to prove that the inequality holds for $k+1$ as well. Since $1+x > 0$, we can multiply both sides of (1) by $1+x$ without the inequality changing sign and obtain

$$(1+x)^{k+1} = (1+x)(1+x)^k \ge (1+x)(1+kx) = 1 + x + kx + kx^2 \ge 1 + (k+1)x$$

since $kx^2 \ge 0$. Thus we got

$$(1+x)^{k+1} \ge 1 + (k+1)x,$$

which is exactly what we needed to show.

In conclusion, We have proven by induction that for any integer $N \ge 0$ and $x > -1$ we have $(1+x)^N \ge 1 + Nx$. This result in knows as *Bernoulli's inequality*.

5) Find the derivative of the real function $f(x) = e^{\left(\frac{x}{x^2+1}\right)}$. At each step, indicate which rule of differentiation is being applied.

Solution. We have

$$\begin{aligned}
f'(x) &\overset{(1)}{=} [e^{\left(\frac{x}{x^2+1}\right)}]' && \text{(Chain Rule)}\\
&\overset{(2)}{=} e^{\left(\frac{x}{x^2+1}\right)} \cdot \left[\frac{x}{x^2+1}\right]' && \text{(Quotient Rule)}\\
&\overset{(3)}{=} e^{\left(\frac{x}{x^2+1}\right)} \cdot \frac{[x]' \cdot (x^2+1) - x \cdot [(x^2+1)]'}{(x^2+1)^2} && \text{(Sum Rule and Basic Derivatives)}\\
&\overset{(4)}{=} e^{\left(\frac{x}{x^2+1}\right)} \cdot \frac{1 \cdot (x^2+1) - x \cdot [(x^2)' + (1)']}{(x^2+1)^2} && \text{(Power Rule and Basic Derivatives)}\\
&\overset{(5)}{=} e^{\left(\frac{x}{x^2+1}\right)} \cdot \frac{x^2+1 - x \cdot (2x+0)}{(x^2+1)^2} && \text{(Tidy up the rational expression)}\\
&\overset{(6)}{=} \frac{e^{\frac{x}{x^2+1}}\,(1-x^2)}{(x^2+1)^2}. && \text{(Final Result)}
\end{aligned}$$

6) Use Implicit Differentiation to obtain an equation of the tangent line to the ellipse $3x^2 + 2y^2 = 11$ at the point $(-1, 2)$.

Solution. We differentiate both sides of the equation with respect to x and express y' from the resulting relation:

$$3x^2 + 2y^2 = 11 \quad \Rightarrow \quad 6x + 4y \cdot y' = 0 \qquad \text{(Power Rule and Chain Rule)}$$

$$y' = \frac{-3x}{2y} \qquad \text{(Rearrange to solve for } y'\text{)}$$

Then consider the point $(-1, 2)$. At $(-1, 2)$, $y' = \frac{-3(-1)}{2(2)} = \frac{3}{4}$. To find the tangent line, we use this value of y' in the formula for the tangent line, i.e.

$$\begin{aligned}
y &= y'(x_0, y_0)(x - x_0) + y_0 && \text{(Use appropriate values for } (x_0, y_0) \text{ and } y'\text{)}\\
y &= \frac{3}{4}(x - (-1)) + 2\\
y &= \frac{3}{4}x + \frac{3}{4} + 2 = \frac{3}{4} \cdot x + \frac{11}{4}\\
y &= \frac{1}{4}(3x + 11).
\end{aligned}$$

7) Suppose f is an odd function and is differentiable everywhere. Prove that for every number $b > 0$, there exists a $c \in (-b, b)$ such that $f'(c) = f(b)/b$.

Solution. Since f is differentiable everywhere, f is continuous on $[-b, b]$ and differentiable on $(-b, b)$. Therefore, by the Mean Value Theorem there exist a $c \in (-b, b)$ such that

$$f'(c) = \frac{f(b) - f(-b)}{b - (-b)} \overset{(*)}{=} \frac{f(b) - (-f(b))}{2b} = \frac{2f(b)}{2b} = \frac{f(b)}{b}.$$

At $(*)$ we used that $f(-b) = -f(b)$ since f is odd.

8) Apply l'Hospital's Rule to find the following limit: $\lim_{x\to 0} \left[\cos(x)\right]^{\frac{1}{\sin(x)}}$.

Solution. To apply l'Hospital's rule, we need to find an indeterminate form: "0/0" or "∞/∞". As it is now, the expression is not in an indeterminate form. Notice however that moving $\cos(x)$ to the exponent by writing

$$[\cos(x)]^{\frac{1}{\sin(x)}} = \exp\left(\frac{1}{\sin(x)}\ln(\cos(x))\right),$$

where $\exp(x) = e^x$ is used to keep things readable, the exponent has an indeterminate form as desired, specifically "0/0". We note that we are only allowed to do this because when x is near 0, $\cos(x)$ is near 1, so the value of $\ln(\cos(x))$ exists (Question: why would it would be problematic if we considered $\ln(\sin(x))$ as x approached zero?). Recall that if f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

[This is Theorem 8 on page 120 of the textbook.] In our case, $f(x) = e^x$ is continuous everywhere, hence, if $\lim_{x\to 0}\frac{\ln(\cos(x))}{\sin(x)}$ exists, then we can compute the limit by the above method. The limit in the exponent can by found using l'Hospital's Rule:

$$\lim_{x\to 0}\frac{\ln(\cos(x))}{\sin(x)} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\frac{-\sin(x)}{\cos(x)}}{\cos(x)} = 0,$$

where the last equality follows by direct substitution. Therefore we have

$$\lim_{x\to 0}[\cos(x)]^{\frac{1}{\sin(x)}} = \exp\left(\lim_{x\to 0}\frac{\ln(\cos(x))}{\sin(x)}\right) = e^0 = 1.$$

9) Determine the extrema (max/min) of the real function $f(x) = xe^{-x^2}$.

Solution. Firstly we write xe^{-x^2} as $\frac{x}{e^{x^2}}$ and apply l'Hospital's Rule to determine the horizontal asymptotes:

$$\lim_{x\to\infty}\frac{x}{e^{x^2}} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{x}{e^{x^2}} \overset{\text{l'H}}{=} \lim_{x\to-\infty}\frac{1}{2xe^{x^2}} = 0.$$

Here we used Basic Derivatives, the Chain Rule and the Power Rule. As the function is continuous on $\mathbb{R}$, this implies that the extreme points (minimum and maximum) can be explicitly found using the First- or Second Derivative Test. Using Product Rule and Chain Rule we calculate:

$$f'(x) = \left(xe^{-x^2}\right)' = (x)'e^{-x^2} + x\left(e^{-x^2}\right)' = e^{-x^2} + xe^{-x^2}(-x^2)' = e^{-x^2} + xe^{-x^2}(-2x) = (1-2x^2)e^{-x^2} \quad (2)$$

and

$$f''(x) = \left((1-2x^2)e^{-x^2}\right)' = (1-2x^2)'e^{-x^2} + \left(e^{-x^2}\right)' = -4xe^{e^{-x^2}} + (1-2x^2)e^{-x^2}(-2x) = 2x(2x^2-3)e^{-x^2}. \quad (3)$$

Our aim is to locate the extrema of $f(x)$, thus we need to know where $f'(x)$ vanishes, that is we need to solve $f'(x) = 0$ for x. Glancing at formula (2) we see that

$$f'(x) = 0 \iff (1-2x^2)e^{-x^2} = 0 \iff x = \pm\frac{1}{\sqrt{2}}$$

since $e^{-x^2} > 0$ for all $x \in \mathbb{R}$. The nature of these critical numbers can be determined by applying one of the derivative tests:

- The *First Derivative Test* means checking if $f'(x)$ changes sign at the critical number. From (2) we see that

$$\begin{cases} f'(x) < 0, & \text{if } x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}, \\ f'(x) > 0, & \text{if } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \end{cases}$$

that is $f'(x)$ changes its sign from negative to positive at $x = -\frac{1}{\sqrt{2}}$ and from positive to negative at $x = \frac{1}{\sqrt{2}}$. Therefore $f(x)$ has a *minimum* at $x = -\frac{1}{\sqrt{2}}$ and a *maximum* at $x = \frac{1}{\sqrt{2}}$ taking the values

$$f(-\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}}e^{-1/2} = -\frac{1}{\sqrt{2e}}, \quad f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}.$$

- The *Second Derivative Test* is done by computing the sign of $f''(x)$ at the critical number. Plugging $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$ into (3) yields

  $$f''(-\frac{1}{\sqrt{2}}) = 2\sqrt{\frac{2}{e}} > 0 \quad \text{and} \quad f''(\frac{1}{\sqrt{2}}) = -2\sqrt{\frac{2}{e}} < 0,$$

  i.e. $f(x)$ has a *minimum* at $x = -\frac{1}{\sqrt{2}}$ and a *maximum* at $x = \frac{1}{\sqrt{2}}$ with the values

  $$f(-\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}}e^{-1/2} = -\frac{1}{\sqrt{2e}}, \quad f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}.$$

In conclusion, the (absolute) extrema of the function $f(x) = xe^{-x^2}$ are $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$.